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AN M-SPACE WHICH IS NOT ISOMORPHIC TO A $C(K)$ SPACE

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ABSTRACT

We give a new example of an $L^1(\mu)$ predual which is not isomorphic to a $C(K)$ space. The space constructed here is a non-separable M-space. By a previous theorem of the author, the separable M-spaces are isomorphic to $C(K)$ spaces.

While Banach spaces whose duals are $L^1(\mu)$ spaces share many of the properties of $C(K)$ spaces, an example was constructed in [2] of an $L^1(\mu)$ predual which is not isomorphic to a $C(K)$ space. It was later proved in [1] that a more restricted class of $L^1(\mu)$ preduals, namely, the separable G-spaces (and in particular separable M-spaces), are isomorphic to $C(K)$ spaces. (Recall that a subspace X of a $C(K)$ space is called an M-space, if there are points $k_\alpha, h_\alpha \in K$ and non-negative numbers τ_α, such that $X = \{f \in C(K): f(k_\alpha) = \tau_\alpha f(h_\alpha)$ for all $\alpha\}$. By a famous result of Kakutani [4], the M-spaces are exactly the closed sublattices of $C(K)$ spaces.) In this paper we show that the separability condition in [1] cannot be dropped. We construct a non-separable M-space which is not isomorphic to a complemented subspace of any $C(K)$ space. The example here is simpler than the one in [2], the latter is however a separable space, and in fact a predual of l_1.

For a Banach space X, let $\lambda(X) = \inf \| T \| \| T^{-1} \| \| P \|$, where the inf is taken over all isomorphisms T of X into a $C(K)$ space, and all possible projections P from $C(K)$ onto TX. Denote by B_{X^*} the unit ball of X^* in its w^*-topology, and consider X canonically as a subspace of $C(B_{X^*})$. By the lemma in [2], $\lambda(X) = \inf \| P \|$, where the inf is taken over all projections P from $C(B_{X^*})$ onto X.

We now describe the example. Let S be the disjoint union of βN and $\beta N - N$ (where βN is the Stone–Cech compactification of the integers.) Denote by K the

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copy of $\beta N - N$ in βN, and by H the disjoint copy of $\beta N - N$, and let $\varphi : K \to H$ be the map that identifies K and H. For a fixed number $0 < \tau < 1$ let $X_\tau = \{f \in C(S): f(k) = \tau f(\varphi(k))$ for all $k \in K\}$. X_τ is an M space, and we shall show that $\lambda(X_\tau) \geqq 1/\tau$. (In fact X_τ is $1/\tau$-isomorphic to $C(\beta N)$ and is thus a $P_{1/\tau}$ space. Hence $\lambda(X_\tau) = 1/\tau$).

The space $X = (\Sigma_n \oplus X_{1/n})_{c_0}$ is an M-space, and $\lambda(X) \geqq \lambda(X_{1/n}) \geqq n$ for all n, i.e. $\lambda(X) = \infty$ and X is not complemented in any $C(K)$ space.

A standard computation gives that X_τ^* can be identified with the measures μ on S that vanish on K, i.e., if we denote by δ_n the point mass at n, then $\mu = \Sigma\, \mu(n)\delta_n + \nu$, where ν is supported on H.

We start with two lemmas.

Lemma 1. *Let ν be a finite measure on some measure space, $\delta > 0$, l, a natural member and $\{A_i\}_{i=1}^N$ measurable sets such that every $l+1$ different A_i's have empty intersection. If $N > \|\nu\| l/\delta$, there is a $1 \leqq j \leqq N$ with $|\nu|(A_j) < \delta$.*

Proof. Let $\{B_k\}$ be the atoms of the field generated by the A_i's. By the condition, each B_k is contained in at most l A_j's. Thus

$$\sum_{j=1}^{N} |\nu|(A_j) \leqq l \sum_k |\nu|(B_k) \leqq l \|\nu\|$$

and since $N > \|\nu\| l/\delta$, there must be a j with $|\nu|(A_j) < \delta$.

Lemma 2. *Let ν_n be measures on $B_{X_\tau^*}$, with $\sup \|\nu_n\| = C < \infty$, and $\varepsilon > 0$. Then there exists an infinite subset J of N, such that for all $j \in J$,*

$$|\nu_j|\{\mu \in B_{X_\tau^*}: \Sigma_{n \in J\, n \neq j} |\mu(n)| > \varepsilon\} < \varepsilon.$$

Proof. Since the ν_n are finite measures, we can find for each n a number $m(n) > n$ such that $|\nu_n|\{\mu \in B_{X_\tau^*}: \Sigma_{k > m(n)} |\mu(k)| > \varepsilon/2\} < \varepsilon/2$.

Let $A_n = \{\mu \in B_{X_\tau^*}: |\mu(n)| > \varepsilon/4\}$. Since each $\mu \in B_{X_\tau^*}$ has norm at most one, every collection of more then $4/\varepsilon$ of the A_n's has empty intersection. By Lemma 1 we can find for each ν_k, a $j \leqq 16C/\varepsilon^2$ such that $|\nu_k|(A_j) \leqq \varepsilon/4$. Thus we can find $j_1 \leqq 16C/\varepsilon^2$ and an infinite set J_1 satisfying

(a) $|\nu_k|(A_{j_1}) \leqq \varepsilon/4$ for all $k \in J_1$,

(b) $k > m(j_1)$ for all $k \in J_1$.

Now let $B_n = \{\mu \in B_{X_\tau^*}: |\mu(n)| > \varepsilon/8\}$. By the same argument as before, there is a $j_2 \in J_1$ and an infinite set $J_2 \subset J_1$ with

(a) $|\nu_k|(B_{j_2}) \leqq \varepsilon/8$ for all $k \in J_2$,

(b) $k > m(j_2)$ for all $k \in J_2$.

Continuing similarly we get for each n a j_{n+1} in J_n and an infinite subset $J_{n+1} \subset J_n$ with

(a) $|\nu_k|\{\mu \in B_{X_\tau^*}: |\mu(j_{n+1})| > \varepsilon/2^{n+2}\} \leqq \varepsilon/2^{n+2}$, for all $k \in J_{n+1}$,

(b) $k > m(j_{n+1})$ for all $k \in J_{n+1}$.

We now take $J = \{j_1, j_2, j_3, \cdots\}$. Then if $j \in J$ we have

$$\Big\{\mu \in B_{X_\tau^*}: \sum_{\substack{n \in J \\ n \neq j}} |\mu(n)| > \varepsilon\Big\} \subset \Big\{\mu : \sum_{n > m(j)} |\mu(n)| > \varepsilon/2\Big\}$$

$$\cup \bigcup_{i<j} \{\mu : |\mu(j_i)| > \varepsilon/2^{i+1}\}$$

and thus

$$|\nu_j| \Big\{\mu : \sum_{\substack{n \in J \\ n \neq j}} |\mu(n)| > \varepsilon\Big\} \leqq \varepsilon/2 + \sum_{i<j} \varepsilon/2^{i+1} < \varepsilon.$$

Proof that $\lambda(X_\tau) \geqq 1/\tau$. We start with the remark that if x^* is any w^*-limit point of $\{\delta_n\}$, then, since every $f \in X_\tau$ is continuous on S, x^* is a point evaluation at some point $k \in K$. Thus $x^*(f) = f(k) = \tau f(\varphi(k))$ for all $f \in X_\tau$, and in particular, $\|x^*\| = \tau$.

Now fix a projection P from $C(B_{X_\tau^*})$ onto X_τ. Given any number $2 > t > 1$, we shall construct an infinite subset M of N, and a function Ψ in $C(B_{X_\tau^*})$ with $\|\Psi\| \leqq t$, such that $\delta_m(P\Psi) \geqq 1/t$ for all $m \in M$. If x^* is any w^*-limit point of $\{\delta_m\}_{m \in M}$, then also $x^*(P\Psi) \geqq 1/t$, but by the remark above $\|x^*\| = \tau$ and hence $1/t \leqq \|\Psi\|\,\|x^*\|\,\|P\| \leqq t\tau\|P\|$, or $\|P\| \geqq 1/t^2\tau$. Since $t > 1$ was arbitrary, we get $\|P\| \geqq 1/\tau$.

Fix $\varepsilon > 0$ such that $1 + \varepsilon(1 + 1/\tau) \leqq t$ and $1 - ((1 + 1/\tau)\|P\| + 3)\varepsilon \geqq 1/t$, and let J be the infinite set obtained by applying Lemma 2 for this ε and $\nu_n = P^*\delta_n$. (We identify $P^*\delta_n \in C(B_{X_\tau^*})^*$ with Baire measures on $B_{X_\tau^*}$ by Riesz's theorem).

For $j \in J$ denote by

$$G_j = \Big\{\mu \in B_{X_\tau^*}: \sum_{\substack{n \in J \\ n \neq j}} |\mu(n)| \leqq \varepsilon\Big\}.$$

For a subset L of N, let $L_H = \varphi(\bar{L} \backslash L)$, i.e., L_H is the copy in H of the limit points of L in βN. Let $\mathcal{M}$ be an uncountable collection of infinite subsets of J such that any pair of them has at most a finite intersection. For each $M \in \mathcal{M}$, the set

$$G(M) = \{\mu \in B_{X_\tau^*}: |\mu(M_H)| > \varepsilon\} = \bigcup_i \bigcup_m \bigcap_{n>m} \{\mu : |\textstyle\int g_n d\mu| > \varepsilon + 1/i\}$$

is a Baire subset of $B_{X_\tau^*}$, where $g_n \in X_\tau$ is given by

$$g(s) = \begin{cases} 1 & s \in M_H \\ \tau & s \in \overline{M\backslash\{1,2,\cdots,n\}} \\ 0 & \text{otherwise} \end{cases}$$

Since $M_H \cap L_H = \emptyset$ for distinct M and L in $\mathcal{M}$, every collection of more then $1/\varepsilon$ of the $G(M)$'s have empty intersection. Since $\mathcal{M}$ is uncountable, we get by Lemma 1 that there is an $M \in \mathcal{M}$ such that $|\nu_n|(G(M)) = 0$ for all $n \in N$. This is the set M we choose.

To define Ψ, let $g \in X_\tau$ be defined by

$$g(s)\begin{cases} 1 & s \in \bar{M} \\ = 1/\tau & s \in M_H \\ 0 & \text{otherwise.} \end{cases}$$

As an element in X_τ, g is also in $C(B_{X_\tau^*})$ and we define for $\mu \in B_{X_\tau^*}$

$$\Psi(\mu) = \begin{cases} g(\mu) & \text{if} \quad |g(\mu)| \leqq t \\ t & \text{if} \quad g(\mu) \geqq t \\ -t & \text{if} \quad g(\mu) \leqq -t. \end{cases}$$

Let us first compute $g(\mu)$ for $\mu \in G_m \backslash G(M)$, $m \in M$.

By the definition of g, $g(\mu) = \Sigma_{n \in M}\, \mu(n) + 1/\tau\mu(M_H)$. But $\mu \notin G(M)$ implies that $|\mu(M_H)| \leqq \varepsilon$, also $\mu \in G_m$ implies that $\Sigma_{n \in M\, n \neq m} |\mu(n)| \leqq \varepsilon$ (recall that $M \subset J$). Hence $g(\mu) = \mu(m) + \eta(\mu)$, where $|\eta(\mu)| \leqq \varepsilon + \varepsilon/\tau$. In particular if $\mu \in G_m \backslash G(M)$, $|g(\mu)| \leqq 1 + \varepsilon(1 + 1/\tau) \leqq t$ and $\Psi(\mu) = g(\mu)$.

Let e_m be the unique function in X_τ such that $e_m(n) = 0$ if $n \neq m$, $e_m(m) = 1$. Since $e_m \in X_\tau$, $Pe_m = e_m$ and thus

$$\delta_m(P\Psi) \geqq \delta_{m_0}(e_m) - |\delta_m(e_m - P\Psi)| = 1 - |P^*\delta_m(e_m - \Psi)|.$$

We now estimate $P^*\delta_m(e_m - \Psi)$ for $m \in M$:

$$\left|\int (\Psi - e_m)dP^*\delta_m\right| \leqq \left|\int_{G_m \backslash G(M)} (\Psi - e_m)dP^*\delta_m\right| + \left|\int_{B_{X_\tau^*} \backslash (G_m \backslash G(M))} (\Psi - e_m)dP^*\delta_m\right|.$$

By the choice of M, $|P^*\delta_m|(G(M)) = 0$, and by the choice of J

$|P^*\delta_m|(B_{X_\tau^*}\backslash G_m) \leqq \varepsilon$. Since also $\|\Psi - e_m\| \leqq t+1 \leqq 3$, the second integral is bounded by 3ε.

On $G_m \backslash G(M)$ we have already computed that $\Psi(\mu) = g(\mu) = \mu(m) + \eta(\mu) = e_m(\mu) + \eta(\mu)$ where $|\eta(\mu)| \leqq \varepsilon(1+1/\tau)$, and thus the first integral is bounded by $\varepsilon(1+1/\tau)\|P\|$.

Hence $\delta_m(P\Psi) \geqq 1 - (1+1/\tau)\varepsilon\|P\| - 3\varepsilon \geqq 1/t$ by the choice of ε. This completes the proof.

REMARK. Notice that X_τ^* contains a τ-norming sequence, namely, the functionals $\{\delta_n\}$. Moreover, τ is the best norming constant of a countable subset of X_τ^*. Indeed, if $\{\mu_n\} \subset B_{X_\tau^*}$, we can find, by using an uncountable collection of finitely intersecting infinite subsets of N, a subset M with $|\mu_n|(M_H) = 0$ for all n. Let $g \in X_\tau$ be defined by

$$g(s) = \begin{cases} 1 & s \in \bar{M} \\ 1/\tau & s \in M_H \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|g\| = 1/\tau$, however, for each n, by the choice of M and the definition of g

$$\left| \int_S g d\mu_n \right| = \left| \int_N g d\mu_n \right| \leqq 1.$$

Thus if $X = (\Sigma \oplus X_{1/n})_{c_0}$, X^* contains a countable total set, but no countable norming set. (The first example of this type was constructed in [3].)

PROBLEM. Let K be a compact Hausdorff space, such that $C(K)^*$ contains a countable total set. Does $C(K)^*$ contain a countable norming set?

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